Branching laws for polynomial endomorphisms in CAR algebra for fermions, uniformly hyperfinite algebras and Cuntz algebras

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Abstract

Previously, we have shown that the CAR algebra for fermions is embedded in the Cuntz algebra $\mathcal{O}_2$ in such a way that the generators are expressed in terms of polynomials in the canonical generators of the latter, and it coincides with the $U(1)$-fixed point subalgebra $\mathcal{A} \equiv \mathcal{O}_2^{U(1)}$ of $\mathcal{O}_2$ for the canonical gauge action. Based on this embedding formula, some properties of $\mathcal{A}$ are studied in detail by restricting those of $\mathcal{O}_2$. Various endomorphisms of $\mathcal{O}_2$, which are defined by polynomials in the canonical generators, are explicitly constructed, and transcribed into those of $\mathcal{A}$. Especially, we investigate branching laws for a certain family of such endomorphisms with respect to four important representations, i.e., the Fock representation, the infinite wedge representation and their duals. These endomorphisms are completely classified by their branching laws. As an application, we show that the reinterpretation of the Fock vacuum as the Dirac vacuum is described in representation theory through a mixture of fermions.

1 Introduction

In previous papers [1, 2, 3, 4], we have presented a recursive construction of the CAR (canonical anticommutation relation) algebra for fermions in terms of the Cuntz algebra $\mathcal{O}_2$ and shown that it may provide us a useful

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tool to study properties of fermion systems by using explicit expressions in terms of generators of the algebra.

Let $\mathcal{A}_0$ be the algebra generated by a_n, a_n^* for $n \in \mathbf{N} \equiv \{1,2,3,\ldots\}$ which satisfy the canonical anticommutation relations (=CAR):

$$a_n a_m^* + a_m^* a_n = \delta_{n,m} I, \quad a_n^* a_m^* + a_m^* a_n^* = a_n a_m + a_m a_n = 0 \quad (n, m \in \mathbf{N}). \tag{1.1}$$

The $*$-algebra $\mathcal{A}_0$ always has unique C*-norm $\|\cdot\|$ and the completion $\mathcal{A}$ of $\mathcal{A}_0$ with respect to $\|\cdot\|$ is called the *CAR algebra* in theory of operator algebras. Let s_1, s_2 be canonical generators of $\mathcal{O}_2$. Then

$$a_1 \equiv s_1 s_2^*, \quad a_n \equiv \sum_{J \in \{1,2\}^{n-1}} s_J s_1 s_2^* \beta_2(s_J)^* \quad (n \geq 2) \tag{1.2}$$

satisfy (1.1) where $s_J \equiv s_{j_1} \cdots s_{j_k}$ for $J = (j_1, \ldots, j_k)$ and β_2 is the automorphism of $\mathcal{O}_2$ defined by $\beta_2(s_i) \equiv (-1)^{i-1} s_i$ for $i = 1, 2$ [1]. Furthermore, $C^*\langle\{a_n \in \mathcal{O}_2 : n \in \mathbf{N}\}\rangle$ coincides with a fixed-point subalgebra $\mathcal{O}_2^{U(1)} \equiv \{x \in \mathcal{O}_2 : \text{for all } z \in U(1), \gamma_z(x) = x\}$ of $\mathcal{O}_2$ for the canonical gauge action γ defined by $\gamma_z(s_i) = z s_i$ ($i = 1, 2$) with $z \in U(1)$. Define the linear map ζ on $\mathcal{O}_2$ by

$$\zeta(x) \equiv s_1 x s_1^* - s_2 x s_2^* \quad (x \in \mathcal{O}_2). \tag{1.3}$$

Then we have $a_n = \zeta(a_{n-1})$ for each $n \geq 2$. In this sense, $\{a_n\}_{n \in \mathbf{N}}$ in (1.2) is called the *recursive fermion system (=RFS)* in $\mathcal{O}_2$. Remark that (1.2) is a finite sum giving a purely algebraic embedding, while it is also a (continuous) embedding of the C*-algebra $\mathcal{A}$ into $\mathcal{O}_2$. In contrast with the so-called boson-fermion correspondence [17, 18], in which the map between bosons and fermions depends on their representations, the generators of fermions are always written by noncommutative homogeneous polynomials ($\in \mathbf{C}[s_1, s_2, s_1^*, s_2^*]$) in the canonical generators of $\mathcal{O}_2$ without use of any representation.

We have also shown [3] that it is possible to generalize this recursive construction to the algebra for the FP ghost fermions in string theory by introducing a $*$-algebra called the pseudo-Cuntz algebra suitable for actions on an indefinite-metric state vector space. We have found that, according to embeddings of the FP ghost algebra into the pseudo-Cuntz algebra with a special attention to the zero-mode operators, unitarily inequivalent representations for the FP ghost are obtained from a single representation of the pseudo-Cuntz algebra.

Based on the embedding formula (1.2), some properties of $\mathcal{A}$ are studied in detail by restricting those of $\mathcal{O}_2$. For this purpose, we start with branching laws of representations of $\mathcal{O}_N$ restricted on $\mathcal{O}_N^{U(1)}$.

Definition 1.1 (Permutative representation of $\mathcal{O}_N$) *Let $s_1, \ldots, s_N$ be canonical generators of $\mathcal{O}_N$.*

(i) *For $J = (j_l)_{l=1}^k \in \{1, \ldots, N\}^k$, $P(J)$ is the class of representations $(\mathcal{H}, \pi)$ of $\mathcal{O}_N$ with a cyclic unit vector $\Omega \in \mathcal{H}$ such that $\pi(s_J)\Omega = \Omega$ and $\{\pi(s_{j_l} \cdots s_{j_k})\Omega\}_{l=1}^k$ is an orthonormal family in $\mathcal{H}$ where $s_J \equiv s_{j_1} \cdots s_{j_k}$. Here, $\{\pi(s_{j_l} \cdots s_{j_k})\Omega\}_{l=1}^k$ is called the cycle of $P(J)$.*

(ii) *Let $\{1, \ldots, N\}^\infty \equiv \{(i_n)_{n\in\mathbf{N}} :$ for any n, $i_n \in \{1, \ldots, N\}\}$. For $J = (j_n)_{n\in\mathbf{N}} \in \{1, \ldots, N\}^\infty$, $P(J)$ is the class of representations $(\mathcal{H}, \pi)$ of $\mathcal{O}_N$ with a cyclic unit vector $\Omega \in \mathcal{H}$ such that $\{\pi((s_{J_{(n)}})^*)\Omega : n \in \mathbf{N}\}$ is an orthonormal family in $\mathcal{H}$ where $J_{(n)} \equiv (j_1, \ldots, j_n)$. Here, $\{\pi((s_{J_{(n)}})^*)\Omega : n \in \mathbf{N}\}$ is called the chain of $P(J)$.*

We simply call that $(\mathcal{H}, \pi)$ is a *cycle (chain)* if there is $J \in \{1, \ldots, N\}^k$ (*resp.* $J \in \{1, \ldots, N\}^\infty$) such that $(\mathcal{H}, \pi)$ belongs to $P(J)$.

Let UHF_N be the uniformly hyperfinite algebra and fix a generating set $\bigcup_{l\geq1}\{E_{K,L}\}_{J,K\in\{1,\ldots,N\}^l}$ of UHF_N such that $\{E_{K,L}\}_{J,K\in\{1,\ldots,N\}^l}$ is a system of matrix units of a unital subalgebra of UHF_N which is isomorphic to $M_{N^l}(\mathbf{C})$ and $E_{K,L} = \sum_{i=1}^N E_{K\cup(i),L\cup(i)}$ for each $K, L \in \{1, \ldots, N\}^l$ and $l \geq 1$ where $K \cup (i) \equiv (k_1, \ldots, k_l, i)$ with $K = (k_1, \ldots, k_l)$. If there is no ambiguity, then $E_{K,L}$ is also denoted as E_{KL}. The following definition depends on the choice of $\bigcup_{l\geq1}\{E_{KL}\}_{J,K\in\{1,\ldots,N\}^l}$.

Definition 1.2 (Permutative representation of UHF_N) *Let $(\mathcal{H}, \pi)$ be a representation of UHF_N.*

(i) *For $J = (j_n)_{n\in\mathbf{N}} \in \{1, \ldots, N\}^\infty$, $P[J]$ is the class of representations $(\mathcal{H}, \pi)$ with a cyclic unit vector $\Omega \in \mathcal{H}$ such that $\pi(E_{J_n,J_n})\Omega = \Omega$ for each $n \geq 1$ where $J_n \equiv (j_1, \ldots, j_n)$.*

(ii) *For $J = (j_n)_{n=1}^k \in \{1, \ldots, N\}^k$, $P[J]$ is the class of representations $(\mathcal{H}, \pi)$ such that $(\mathcal{H}, \pi)$ is $P[J^\infty]$.*

In both Definition 1.1 and 1.2, we call Ω the *GP (Generalized Permutative) vector* of $(\mathcal{H}, \pi)$.

Identifying UHF_N with $\mathcal{O}_N^{U(1)}$ by the embedding Φ_{UHF_N} of UHF_N into $\mathcal{O}_N$ as

$$\Phi_{UHF_N}(E_{JK}) \equiv s_J s_K^* \quad (J, K \in \{1, \ldots, N\}^n,\, n \geq 1), \tag{1.4}$$

we show explicit branching laws of permutative representations of $\mathcal{O}_N$ restricted on UHF_N. In the case that a representation $(\mathcal{H},\pi)$ of $\mathcal{O}_N$ is $P(J)$, we denote the restriction $(\mathcal{H},\pi|_{UHF_N})$ by $P(J)|_{UHF_N}$ for simplicity of description.

Theorem 1.3 *For $J=(j_1,\ldots,j_l)\in\{1,\ldots,N\}^l$ and $K=(k_n)_{n\in\mathbf{N}}\in\{1,\ldots,N\}^\infty$, the following irreducible decompositions hold:*

$$P(J)|_{UHF_N}=\bigoplus_{\sigma\in\mathbf{Z}_l}P[\sigma J],\quad P(K)|_{UHF_N}=\bigoplus_{\eta\in\mathbf{Z}}P[\eta K],\tag{1.5}$$

where $\sigma J\equiv(j_{\sigma(1)},\ldots,j_{\sigma(l)})$ and $\eta K=(k_n^{'})_{n\in\mathbf{N}}$ is defined by $k_n^{'}\equiv k_{\eta(n)}$ for $\eta(n)\geq 1$, $k_n^{'}\equiv 1$ for $\eta(n)\leq 0$.

In [6], this result is described in terms of the "atom" of a permutative representation. The decomposition of the chain is a special case of [7].

Let $\mathfrak{S}_{N,l}$ be the set of all permutations on the set $\{1,\ldots,N\}^l$. For $\sigma\in\mathfrak{S}_{N,l}$, define the endomorphism ψ_σ of $\mathcal{O}_N$ by

$$\psi_\sigma(s_i)\equiv u_\sigma s_i\quad(i=1,\ldots,N),\tag{1.6}$$

where $u_\sigma\equiv\sum_{J\in\{1,\ldots,N\}^l}s_{\sigma(J)}(s_J)^*$. It should be noted that ψ_σ commutes with the $U(1)$-gauge action on $\mathcal{O}_N$ for any σ. In [13], it is shown the following: Numbers of unitary equivalence classes of elements in $E_{2,2}\equiv\{\psi_\sigma:\sigma\in\mathfrak{S}_{2,2}\}$ is 16. The group $G_2\equiv\mathrm{Aut}\mathcal{O}_2\cap E_{2,2}$ is isomorphic to the Klein's four-group, which consists of two outer and two inner automorphisms. The set $E_{2,2}\setminus G_2$ consists of 10 irreducible and 10 reducible endomorphisms and numbers of their equivalence classes are 5 and 9, respectively. Especially, (the class of) the canonical endomorphism of $\mathcal{O}_2$ belongs to the set of reducible classes in $E_{2,2}$. Now, we consider the restriction of $E_{2,2}$ to UHF_2. Since $\rho|_{UHF_2}$ is also an endomorphism of UHF_2 for each $\rho\in E_{2,2}$, its properties are obtained as follows:

Theorem 1.4 *Define $UE_{2,2}\equiv\{\rho|_{UHF_2}:\rho\in E_{2,2}\}$.*

(i) *The cardinarity of $UE_{2,2}$ equals 20 and the number of unitary equivalence classes of elements in $UE_{2,2}$ is 12.*

(ii) *The subgroup $UG_2\equiv\mathrm{Aut}UHF_2\cap UE_{2,2}$ of $\mathrm{Aut}UHF_2$ is also isomorphic to the Klein's four-group, which consists of two outer and two inner automorphisms of UHF_2.*

(iii) *The set of all equivalence classes in* $UE_{2,2} \setminus UG_2$ *consists of* 4 *irreducibles and* 6 *reducibles.*

With the embedding Φ_{CAR} of the CAR algebra $\mathcal{A}$ into $\mathcal{O}_2$ defined as (1.2), and with the identification $\mathcal{A}$ and UHF_2 by a map Ψ from $\mathcal{A}$ onto UHF_2,

$$\Psi(a_1) \equiv E_{12}, \quad \Psi(a_n) \equiv \sum_{J\in\{1,2\}^{n-1}} (-1)^{n_2(J)} E_{J\cup(1),J\cup(2)} \quad (n \geq 2), \tag{1.7}$$

where $n_2(J) \equiv \sum_{i=1}^{k}(j_i - 1)$ for $J = (j_1, \ldots, j_k)$, we obtain a relation $\Phi_{CAR} = \Phi_{UHF_2} \circ \Psi$ among Ψ, Φ_{CAR} and Φ_{UHF_2} in (1.4). Hence, according to these maps, we may regard $\mathcal{A}$ and UHF_2 as the same subalgebra of $\mathcal{O}_2$. Restrictions of representations and branching laws are also described by such identifications.

As an application of branching laws on UHF_2, we show the following: Let $(\mathcal{H}, \pi)$ be the Fock representation of $\mathcal{A}$ with the vacuum Ω. We identify $\mathcal{A}$ with $\pi(\mathcal{A})$. Then, we have $a_n\Omega = 0$ for each $n \geq 1$. Let $\mathbf{Z} + 1/2 \equiv \{n + 1/2 : n \in \mathbf{Z}$ and $\mathbf{Z}_{\geq} + 1/2 \equiv \{x \in \mathbf{Z} + 1/2 : x > 0\}$. Define a mixture $\{b_k\}_{k\in\mathbf{Z}+1/2}$ of fermions by

$$\left\{\begin{array}{ll} b_k \equiv & (-1)^{k-1/2}(a_1a_1^*a_{2k+2}^* + a_1^*a_1a_{2k+2}), \\ & \\ b_{-k} \equiv & (-1)^{k-1/2}(a_1a_1^*a_{2k+1} - a_1^*a_1a_{2k+1}^*) \end{array}\right. \quad (k \in \mathbf{Z}_{\geq} + 1/2). \tag{1.8}$$

Then, we obtain $b_kb_l + b_lb_k = 0$, $b_kb_l^* + b_l^*b_k = \delta_{kl}I$ for each k, l, and

$$b_k\Omega = (-1)^{k-1/2}a_{2k+2}^*\Omega, \quad b_{-k}^*\Omega = (-1)^{k-1/2}a_{2k+1}^*\Omega, \quad b_k^*\Omega = b_{-k}\Omega = 0 \tag{1.9}$$

for $k > 0$. It is shown that the vectors

$$b_{k_1}\cdots b_{k_n}b_{-l_1}^*\cdots b_{-l_m}^*\Omega \quad (k_1, \ldots, k_n, l_1, \ldots, l_m \in \mathbf{Z}_{\geq} + 1/2) \tag{1.10}$$

span the infinite wedge representation of $\mathcal{A}$ with the Dirac vacuum (or the vacuum of the infinite wedge) Ω [17, 18]. In this way, it becomes possible to reinterpret the Fock vacuum as the Dirac vacuum. On the other hand, for $\Omega^* \equiv a_1^*\Omega$, the following equations hold:

$$b_{-k}\Omega^* = (-1)^{k-1/2}a_{2k+1}^*\Omega^*, \quad b_k^*\Omega^* = (-1)^{k-1/2}a_{2k+2}^*\Omega^*, \quad b_k\Omega^* = b_{-k}^*\Omega^* = 0. \tag{1.11}$$

Hence, the vectors

$$b_{-k_1}\cdots b_{-k_n}b_{l_1}^*\cdots b_{l_m}^*\Omega^* \quad (k_1, \ldots, k_n, l_1, \ldots, l_m \in \mathbf{Z}_{\geq} + 1/2) \tag{1.12}$$

span the dual infinite wedge representation of $\mathcal{A}$ with the dual Dirac vacuum Ω^*. In consequence, through the mixture (1.8) of fermions, the Fock representation is transformed to the direct sum of the infinite wedge representation and its dual:

$$\textit{Fock} \quad \overset{\textit{mixture}}{\Rightarrow} \quad \textit{Infinite Wedge} \oplus \textit{Dual Infinite Wedge}. \tag{1.13}$$

The present paper is organized as follows: We show properties of permutative representations and permutative endomorphisms of the Cuntz algebras and the uniformly hyperfinite algebras in § 2. In § 2.2, Theorem 1.3 is proved. We treat the second order permutative endomorphisms and their branching laws in § 3. In § 3.2, Theorem 1.4 is proved. We apply these results to the case of fermions in § 4. In § 4.2, we explain (1.8) and its meaning in the representation theory.

2 Branching laws on $\mathcal{O}_N$ and UHF_N

In this paper, any representation and endomorphism are assumed unital and $*$-preserving.

2.1 On $\mathcal{O}_N$

For $N \geq 2$, let $\mathcal{O}_N$ be the *Cuntz algebra* [8], that is, a C*-algebra which is universally generated by generators $s_1, \ldots, s_N$ satisfying $s_i^* s_j = \delta_{ij} I$ for $i, j = 1, \ldots, N$ and $s_1 s_1^* + \cdots + s_N s_N^* = I$.

We review results of permutative representations [6, 9, 10]. $(\mathcal{H}, \pi)$ is a *permutative representation* of $\mathcal{O}_N$ if there is a complete orthonormal basis $\{e_n\}_{n\in\Lambda}$ of $\mathcal{H}$ and a family $f = \{f_i\}_{i=1}^N$ of maps on Λ such that $\pi(s_i)e_n = e_{f_i(n)}$ for each $n \in \Lambda$ and $i = 1, \ldots, N$. Any permutative representation is uniquely decomposed into cyclic permutative representations up to unitary equivalence. For any J, $P(J)$ contains only one unitary equivalence class. Any cyclic permutative representation is equivalent to $P(J)$ for a certain $J \in \{1, \ldots, N\}^\# \equiv \coprod_{k\geq 1}\{1, \ldots, N\}^k \sqcup \{1, \ldots, N\}^\infty$.

We prepare several notions of multiintegers. Define $\{1, \ldots, N\}_1^* \equiv \coprod_{k\geq 1}\{1, \ldots, N\}^k$ and $\{1, \ldots, N\}^* \equiv \coprod_{k\geq 0}\{1, \ldots, N\}^k$, $\{1, \ldots, N\}^0 \equiv \{0\}$. The *length* $|J|$ of $J \in \{1, \ldots, N\}^\#$ is defined by $|J| \equiv k$ for $J \in \{1, \ldots, N\}^k$. For $J_1, J_2 \in \{1, \ldots, N\}^*$ and $J_3 \in \{1, \ldots, N\}^\infty$, $J_1 \cup J_2 \equiv (j_1, \ldots, j_k, j_1^{'}, \ldots, j_l^{'})$, $J_1 \cup J_3 \equiv (j_1, \ldots, j_k, j_1^{''}, j_2^{''}, \ldots)$ for $J_1 = (j_a)_{a=1}^k$, $J_2 = (j_b^{'})_{b=1}^l$ and $J_3 = (j_n^{''})_{n\in\mathbf{N}}$. Especially, we define $J \cup (0) = (0) \cup J = J$ for convention. For $J \in \{1, \ldots, N\}^*$ and $k \geq 2$, $J^k \equiv J \cup \cdots \cup J$ (k times). $J \in \{1, \ldots, N\}_1^*$ is *periodic*

if there exist an integer $m \geq 2$ and a multiintegers $J_0 \in \{1,\ldots,N\}_1^*$ such that $J = (J_0)^m$. For $J_1, J_2 \in \{1,\ldots,N\}_1^*$, $J_1 \sim J_2$ if $J_1, J_2 \in \{1,\ldots,N\}^k$ $(k \geq 1)$ and $J_2 = (j_p,\ldots,j_k,j_1,\ldots,j_{p-1})$ $(1 \leq p \leq k)$ with $J_1 \equiv (j_1,\ldots,j_k)$. For $(J,z),(J^{'},z^{'}) \in \{1,\ldots,N\}_1^* \times U(1)$, $(J,z) \sim (J^{'},z^{'})$ if $J \sim J^{'}$ and $z = z^{'}$. For $J_1 = (j_l)_{l=1}^k, J_2 = (j_l^{'})_{l=1}^k$, $J_1 \prec J_2$ if $\sum_{l=1}^k (j_l^{'} - j_l)N^{k-l} \geq 0$. Especially, any element in $\{1,\ldots,N\}$ is nonperiodic. $J \in \{1,\ldots,N\}^\infty$ is *eventually periodic* if there are $J_0, J_1 \in \{1,\ldots,N\}_1^*$ such that $J = J_0 \cup (J_1)^\infty$. For $J_1, J_2 \in \{1,\ldots,N\}^\infty$, $J_1 \sim J_2$ if there exist $J_3, J_4 \in \{1,\ldots,N\}^*$ and $J_5 \in \{1,\ldots,N\}^\infty$ such that $J_1 = J_3 \cup J_5$ and $J_2 = J_4 \cup J_5$.

Next, we introduce a representation of $\mathcal{O}_N$ which is not a permutative one.

Definition 2.1 *$GP(\pm)$ is the class of representations $(\mathcal{H},\pi)$ with a cyclic unit vector $\Omega \in \mathcal{H}$ such that $\pi(s_1 \pm s_2)\Omega = \sqrt{2}\Omega$.*

For $(\mathcal{H},\pi)$ of $P(J)$ and $\rho \in \mathrm{End}\mathcal{O}_N$, we denote $(\mathcal{H},\pi\circ\rho)$ by $P(J)\circ\rho$ for simplicity of description.

Theorem 2.2 *Define $P(J;z^k) \equiv P(J)\circ\gamma_z$ for $z \in U(1)$ and $J \in \{1,\ldots,N\}^k$.*

(i) *For $(J,z) \in \{1,\ldots,N\}_1^* \times U(1)$, $P(J;z)$ is irreducible if and only if J is nonperiodic. For $J \in \{1,\ldots,N\}^\infty$, $P(J)$ is irreducible if and only if J is not eventually periodic.*

(ii) *For $(J_1,z_1),(J_2,z_2) \in \{1,\ldots,N\}_1^* \times U(1)$, $P(J_1;z_1) \sim P(J_2;z_2)$ if and only if $(J_1,z_1) \sim (J_2,z_2)$. For $J_1, J_2 \in \{1,\ldots,N\}^\infty$, $P(J_1) \sim P(J_2)$ if and only if $J_1 \sim J_2$.*

(iii) *For $J \in \{1,\ldots,N\}_1^*$ and $l \geq 1$, $P(J^l) = P(J;\xi_1) \oplus \cdots \oplus P(J;\xi_l)$ where $\xi_n \equiv e^{2\pi\sqrt{-1}(n-1)/l}$. This decomposition is unique up to unitary equivalence. Especially, if J is nonperiodic, then this is multiplicity-free.*

(iv) *$GP(\pm)$ contains only one unitary equivalence class and it is irreducible. Furthermore, $GP(+)$ and $GP(-)$ are not equivalent.*

Proof. (i)-(iii) are proved in [6, 9, 10, 12].
(iv) Define $\phi \in \mathrm{Aut}\mathcal{O}_N$ by $\phi(s_1) \equiv (s_1+s_2)/\sqrt{2}$, $\phi(s_2) \equiv (s_1-s_2)/\sqrt{2}$ and $\phi(s_i) \equiv s_i$ for each $i = 3,\ldots,N$. For any representation $(\mathcal{H},\pi)$, $(\pi\circ\phi^{-1})(s_1+s_2) = \sqrt{2}\pi(s_1)$ and $(\pi\circ\phi^{-1})(s_1-s_2) = \sqrt{2}\pi(s_2)$. This implies that $P(1)\circ\phi^{-1} = GP(+)$ and $P(2)\circ\phi^{-1} = GP(-)$. Therefore, from (ii) and (iii), the statements are verified. ∎

Hereafter, we denote a representation $(\mathcal{H},\pi)$ of $\mathcal{O}_N$ by π for simplicity of description.

Theorem 2.3 *For ψ_σ in (1.6), the following holds:*

(i) *If π is a permutative representation, then $\pi\circ\psi_\sigma$ is also a permutative representation.*

(ii) *If π is $P(J)$ for $J\in\{1,\ldots,N\}^{\#}$ and $\sigma\in\mathfrak{S}_{N,l}$, then there exist multi-integers $J_1,\ldots,J_M\in\{1,\ldots,N\}^{\#}$ and subrepresentations $\pi_1,\ldots,\pi_M$ of $\pi\circ\psi_\sigma$ such that*

$$\pi\circ\psi_\sigma=\pi_1\oplus\cdots\oplus\pi_M \tag{2.1}$$

with π_i being $P(J_i)$ for $i=1,\ldots,M$, and $1\leq M\leq N^{l-1}$.

(iii) *In (ii), if $J\in\{1,\ldots,N\}^k$ ($J\in\{1,\ldots,N\}^\infty$), then $J_i\in\coprod_{a=1}^{N^{l-1}}\{1,\ldots,N\}^{ak}$ (resp. $J_i\in\{1,\ldots,N\}^\infty$) for $i=1,\ldots,M$.*

Proof. See Theorem 1.3 of [15]. ∎

The endomorphism ψ_σ in (1.6) is called the *permutative endomorphism* of $\mathcal{O}_N$ associated with σ.

From the uniqueness of decomposition of the permutative representation, the rhs in (2.1) is unique up to unitary equivalence. Then (2.1) can be rewritten as follows:

$$P(J)\circ\psi_\sigma=P(J_1)\oplus\cdots\oplus P(J_M). \tag{2.2}$$

We call (2.2) the *branching law* for ψ_σ with respect to $P(J)$. The branching law for ψ_σ is unique up to unitary equivalence of ψ_σ. From contraposition to this result, we distinguish two equivalence classes of endomorphisms in §3.

2.2 On UHF_N

Lemma 2.4 *For $J\in\{1,\ldots,N\}^\infty$, let ω be the state of UHF_N defined by*

$$\omega(E_{K,L})\equiv 0\quad(K\neq L),\quad \omega(E_{K,K})\equiv\delta_{K,J_n}\quad(|K|=n). \tag{2.3}$$

Then the GNS representation of UHF_N by ω is $P[J]$ in Definition 1.2.

Proof. Let π be $P[J]$ with the GP vector Ω. Define the state ρ of UHF_N by $\rho \equiv < \Omega|\pi(\cdot)\Omega >$. For ω in (2.3), we can verify that $\rho = \omega$. Because of the cyclicity of Ω and the uniqueness of the GNS representation, the statement holds. ∎

For $J = (j_n)_{n\in\mathbf{N}}, J^{'} = (j^{'}_n)_{n\in\mathbf{N}} \in \{1,\ldots,N\}^{\infty}$, $J \approx J^{'}$ if there exists an integer $n_0 \geq 1$ such that $j_n = j^{'}_n$ for each $n \geq n_0$.

Proposition 2.5

(i) *For any $J \in \{1,\ldots,N\}^{\#}$, $P[J]$ contains only one unitary equivalence class.*

(ii) *For any $J \in \{1,\ldots,N\}^{\#}$, $P[J]$ is irreducible.*

(iii) *For $J, J^{'} \in \{1,\ldots,N\}^{\infty}$, $P[J] \sim P[J^{'}]$ if and only if $J \approx J^{'}$.*

(iv) *For $J, J^{'} \in \{1,\ldots,N\}^{*}_{1}$, $P[J] \sim P[J^{'}]$ if and only if $J = J^{'}$.*

Proof. (i) By Lemma 2.4, the statement holds.
(ii) Let $\varepsilon_1,\ldots,\varepsilon_N$ be the standard basis of $\mathbf{C}^N$. Then, we see that the state ω in (2.3) equals to the state $F^{\phi}_{(0)}$ in Definition 2.3 of [7] for $\phi = (\varepsilon_{j_n})_{n\in\mathbf{N}}$. Because $F^{\phi}_{(0)}$ is pure, the statement holds.
(iii) Applying Theorem 2.5 in [7] to $P[J]$, the assertion holds.
(iv) From (iii) and the definition of $P[J]$, we have $P[J] \sim P[J^{'}]$ if and only if $J^{\infty} \approx (J^{'})^{\infty}$. This is equivalent with $J = J^{'}$. ∎

Proof of Theorem 1.3. Let $(\mathcal{H},\pi)$ be $P(J)$ of $\mathcal{O}_N$ with the GP vector Ω. Here we denote $\pi(s_i)$ by s_i for simplicity of description. In both decomposition formulae in the statement, the irreducibility of each component follows by Proposition 2.5 (ii).

Assume that $J = (j_1,\ldots,j_l)$. From $UHF_N = \mathcal{O}_N^{U(1)}$, we obtain $P(J;z)|_{UHF_N} = P(J)|_{UHF_N}$ for any $z \in U(1)$. Hence, it is sufficient to show that $P(J)|_{UHF_N} = \bigoplus_{\sigma\in\mathbf{Z}_l} P[\sigma J]$. Define V_i by the completion of $V_{i,0} \equiv \pi(UHF_N)e_i$ and $e_i \equiv s_{j_i}\cdots s_{j_l}\Omega$ for $i = 1,\ldots,l$. Now, we show that $\mathcal{H} = V_1 \oplus \cdots \oplus V_l$. By setting $E_{LM} \equiv s_L s_M^*$ for $L, M \in \{1,\ldots,N\}^a$, we see that if $i \neq j$, then $< E_{LM}e_i|E_{L^{'}M^{'}}e_j >= 0$ for each $L, M \in \{1,\ldots,N\}^a$ and $L^{'}, M^{'} \in \{1,\ldots,N\}^b$. Therefore V_i and V_j are orthogonal for $i \neq j$. On the other hand, if $|L| = ln + i$ for $n \geq 0$ and $i = 0,1,\ldots,l-1$, then $s_L\Omega = s_L(s_{j_{l-i+1}}\cdots s_{j_l}(s_J)^n)^* e_{l-i+1}$ and $s_L(s_{j_{l-i+1}}\cdots s_{j_l}(s_J)^n)^* \in UHF_N$. Hence, we have $s_L\Omega \in V_i$. Because $\mathrm{Lin}\langle\{s_L\Omega : L \in \{1,\ldots,N\}^*\}\rangle$ is dense in $\mathcal{H}$, we

have $\mathcal{H} = V_1 \oplus \cdots \oplus V_l$ as a UHF_N-module. Define a state $\omega_i \equiv < e_i|\pi(\cdot)e_i >$ of UHF_N for $i = 1, \ldots, l$ and $\sigma_i J \equiv (j_i, \ldots, j_l, j_1, \ldots, j_{i-1})$. Then ω_i satisfies (2.3) with respect to $(\sigma_i J)^\infty$. Because the restriction $\pi^{[i]}$ of $\pi|_{UHF_N}$ on V_i is equivalent to the GNS representation by ω_i, we have $\pi^{[i]}$ is $P[\sigma_i J]$. Therefore, the first formula holds.

Assume that $K = (k_n)_{n\in\mathbf{N}}$. Define $e_n \equiv s_{k_n}^* \cdots s_{k_1}^* \Omega$, $e_{-n} \equiv s_1^n \Omega$ for $n \geq 1$ and $e_0 \equiv \Omega$. Define V_n by the completion of $V_{n,0} \equiv \pi(UHF_N)e_n$ for $n \in \mathbf{Z}$. Then V_n and V_m are orthogonal for $n \neq m$. Let $L \in \{1, \ldots, N\}^a$ and $n \in \mathbf{Z}$. For $n \geq l$, we have $s_L e_n = s_L s_{k_n}^* \cdots s_{k_{n-a+1}}^* e_{n-a} \in V_{n-a}$. For $1 \leq n < a$, we have $s_L e_n = s_L s_{K_n}^* (s_1^*)^{a-n} e_{n-a} \in V_{n-a}$. For $n \leq 0$, we have $s_L e_n = s_L (s_1^*)^a e_{n-a} \in V_{n-a}$. Hence, we obtain that $s_L e_n \in \bigoplus_{m\in\mathbf{Z}} V_m$ for each n and L. Because $\mathrm{Lin}\langle\{s_L e_n : L \in \{1, \ldots, N\}^*, n \in \mathbf{Z}\}\rangle$ is dense in $\mathcal{H}$, we have $\mathcal{H} = \bigoplus_{n\in\mathbf{Z}} V_n$. Define a state ω_n of UHF_N by $\omega_n \equiv < e_n|\pi(\cdot)e_n >$, and $\sigma_n(m) \equiv m + n$ for $n, m \in \mathbf{Z}$. Then ω_n satisfies (2.3) with respect to $\sigma_n K$. Likewise in the case of J, the second formula holds. ∎

In order to classify endomorphisms of UHF_N, we introduce two representations as follows:

Definition 2.6 *$GP[\pm]$ is the class of representations $(\mathcal{H}, \pi)$ of UHF_N with a cyclic vector Ω such that $\pi(F_{n,\pm})\Omega = \Omega$ for each $n \geq 1$, where $F_{n,\pm} \in UHF_N$ is defined by*

$$F_{n,+} \equiv 2^{-n} \sum_{J,K\in\{1,2\}^n} E_{JK}, \quad F_{n,-} \equiv 2^{-n} \sum_{J,K\in\{1,2\}^n} (-1)^{\|J-K\|} E_{JK} \tag{2.4}$$

with $\|J - K\| \equiv \sum_{i=1}^n (j_i - k_i)$ for $J = (j_i)_{i=1}^n$ and $K = (k_i)_{i=1}^n$. We call Ω the GP vector of $(\mathcal{H}, \pi)$.

Proposition 2.7

(i) *If J is nonperiodic (not eventually periodic), then the case (i) (respectively (ii)) in Theorem 1.3 is multiplicity-free.*

(ii) $GP(\pm)|_{UHF_N} = GP[\pm]$.

(iii) *$GP[\pm]$ contains only one unitary equivalence class and it is irreducible. $GP[+]$ and $GP[-]$ are not equivalent.*

Proof. (i) This holds as a result of Proposition 2.5 (iii) and (iv).
(ii) In the proof of Theorem 2.2 (v), we see that $\phi|_{UHF_N} \in \mathrm{Aut}UHF_N$ and $P[1] \circ \phi^{-1} = GP[+]$. Hence, we have

$$GP[+] \circ \phi = P[1] = P(1)|_{UHF_N} = (GP(+) \circ \phi)|_{UHF_N}. \tag{2.5}$$

From this, we obtain $GP[+] = GP(+)|_{UHF_N}$. In a similar way, we obtain $GP[-] = GP(-)|_{UHF_N}$.
(iii) Because we have $P[1] \circ \phi^{-1} = GP[+]$ and $P[2] \circ \phi^{-1} = GP[-]$, the statements hold. ∎

3 Second order permutative endomorphisms

In order to classify endomorphisms of C*-algebras, we prepare several notions for their properties. Let End$\mathcal{A}$ be the set of all unital $*$-endomorphisms of a unital $*$-algebra $\mathcal{A}$ and $\rho, \rho_1, \rho_2 \in \mathrm{End}\mathcal{A}$. We state that ρ is *proper* if $\rho(\mathcal{A}) \neq \mathcal{A}$; ρ is *irreducible* if $\rho(\mathcal{A})' \cap \mathcal{A} = \mathbf{C}I$; ρ is *reducible* if ρ is not irreducible; ρ_1 and ρ_2 are *equivalent* ($\rho_1 \sim \rho_2$) if there exists a unitary $u \in \mathcal{A}$ such that $\rho_2 = \mathrm{Ad}u \circ \rho_1$.

Then the following holds. If ρ_1 is proper and ρ_2 is not, then $\rho_1 \not\sim \rho_2$. If ρ_1 is irreducible and ρ_2 is not, then $\rho_1 \not\sim \rho_2$. Any automorphism is irreducible and not proper. If $\mathcal{A}$ is simple, then ρ is an automorphism if and only if ρ is not proper. Let Rep$\mathcal{A}$ be the class of all unital $*$-representations of $\mathcal{A}$. If $\mathcal{A}$ is simple and there is $\pi \in \mathrm{Rep}\mathcal{A}$ such that both π and $\pi \circ \rho$ are irreducible, then ρ is irreducible. If there exists $\pi \in \mathrm{Rep}\mathcal{A}$ such that $\pi \circ \rho_1 \not\sim \pi \circ \rho_2$, then $\rho_1 \not\sim \rho_2$. If there exists $\pi \in \mathrm{Rep}\mathcal{A}$ such that π is irreducible and $\pi \circ \rho$ is not, then ρ is proper. If $\mathcal{B}$ is a subalgebra of $\mathcal{A}$ and $\rho \in \mathrm{End}\mathcal{A}$ satisfies $\rho|_{\mathcal{B}} \in \mathrm{End}\mathcal{B}$, then $(\pi \circ \rho)|_{\mathcal{B}} = (\pi|_{\mathcal{B}}) \circ (\rho|_{\mathcal{B}})$ for any $\pi \in \mathrm{Rep}\mathcal{A}$.

3.1 On $\mathcal{O}_2$

In order to derive branching laws on UHF_2, we review the corresponding results on $\mathcal{O}_2$. Recall ψ_σ for $\sigma \in \mathfrak{S}_{N,l}$ in (1.6). For $l = 2$, we call ψ_σ the *second order permutative endomorphisms* of $\mathcal{O}_N$ by σ. In [13], we show the complete classification of unitary equivalence classes of the second order permutative endomorphisms of $\mathcal{O}_2$ by using their branching laws. By using a map $\lambda : \{1,2,3,4\} \to \{(1,1),(1,2),(2,1),(2,2)\}$ defined by $\lambda(1) = (1,1)$, $\lambda(2) = (1,2)$, $\lambda(3) = (2,1)$, $\lambda(4) = (2,2)$, we identify $\sigma \in \mathfrak{S}_{2,2}$ and $\lambda^{-1} \circ \sigma \circ \lambda \in \mathfrak{S}_4$, and use a notation

$$E_{2,2} = \{\psi_\sigma : \sigma \in \mathfrak{S}_4 (\cong \mathfrak{S}_{2,2})\}. \tag{3.1}$$

Their properties are summarized in Table 1 (see Table I in [13]).

Table 1. Elements in $E_{2,2}$.

ψ_σ	$\psi_\sigma(s_1)$	$\psi_\sigma(s_2)$	property	$\mathrm{Ad}u \circ \psi_\sigma$
ψ_{id}	s_1	s_2	*inn.aut*	$\psi_{(14)(23)}$
ψ_{12}	$s_{12,1}+s_{11,2}$	s_2	*irr.end*	ψ_{1324}
ψ_{13}	$s_{21,1}+s_{12,2}$	$s_{11,1}+s_{22,2}$	*irr.end*	ψ_{1432}
ψ_{14}	$s_{22,1}+s_{12,2}$	$s_{21,1}+s_{11,2}$	*red.end*	ψ_{14}
ψ_{23}	$s_{11,1}+s_{21,2}$	$s_{12,1}+s_{22,2}$	*red.end*	ψ_{23}
ψ_{24}	$s_{11,1}+s_{22,2}$	$s_{21,1}+s_{12,2}$	*irr.end*	ψ_{1234}
ψ_{34}	s_1	$s_{22,1}+s_{21,2}$	*irr.end*	ψ_{1423}
ψ_{123}	$s_{12,1}+s_{21,2}$	$s_{11,1}+s_{22,2}$	*red.end*	ψ_{243}
ψ_{132}	$s_{21,1}+s_{11,2}$	$s_{12,1}+s_{22,2}$	*red.end*	ψ_{132}
ψ_{124}	$s_{12,1}+s_{22,2}$	$s_{21,1}+s_{11,2}$	*red.end*	ψ_{124}
ψ_{142}	$s_{22,1}+s_{11,2}$	$s_{21,1}+s_{12,2}$	*irr.end*	ψ_{134}
ψ_{134}	$s_{21,1}+s_{12,2}$	$s_{22,1}+s_{11,2}$	*irr.end*	ψ_{142}
ψ_{143}	$s_{22,1}+s_{12,2}$	$s_{11,1}+s_{21,2}$	*red.end*	ψ_{143}
ψ_{234}	$s_{11,1}+s_{21,2}$	$s_{22,1}+s_{12,2}$	*red.end*	ψ_{234}
ψ_{243}	$s_{11,1}+s_{22,2}$	$s_{12,1}+s_{21,2}$	*red.end*	ψ_{123}
ψ_{1234}	$s_{12,1}+s_{21,2}$	$s_{22,1}+s_{11,2}$	*irr.end*	ψ_{24}
ψ_{1243}	$s_{12,1}+s_{22,2}$	$s_{11,1}+s_{21,2}$	*red.end*	ψ_{1243}
ψ_{1324}	s_2	$s_{12,1}+s_{11,2}$	*irr.end*	ψ_{12}
ψ_{1342}	$s_{21,1}+s_{11,2}$	$s_{22,1}+s_{12,2}$	*red.end*	ψ_{1342}
ψ_{1423}	$s_{22,1}+s_{21,2}$	s_1	*irr.end*	ψ_{34}
ψ_{1432}	$s_{22,1}+s_{11,2}$	$s_{12,1}+s_{21,2}$	*irr.end*	ψ_{13}
$\psi_{(12)(34)}$	$s_{12,1}+s_{11,2}$	$s_{22,1}+s_{21,2}$	*out.aut*	$\psi_{(13)(24)}$
$\psi_{(13)(24)}$	s_2	s_1	*out.aut*	$\psi_{(12)(34)}$
$\psi_{(14)(23)}$	$s_{22,1}+s_{21,2}$	$s_{12,1}+s_{11,2}$	*inn.aut*	ψ_{id}

The symbols are defined as $s_{ij,k} \equiv s_i s_j s_k^*$ for $i, j, k = 1, 2$, and "inn.aut", "out.aut", "irr.end" and "red.end" mean an inner automorphism, an outer automorphism, a proper irreducible endomorphism and a reducible endomorphism, respectively, and $u \equiv s_1 s_2^* + s_2 s_1^*$.

For $\zeta = (\zeta_1, \zeta_2) \in \mathcal{O}_2 \times \mathcal{O}_2$ and $\varphi_1, \varphi_2 \in \mathrm{End}\mathcal{O}_2$, define a linear $*$-preserving transformation $\varphi_1 +_\zeta \varphi_2$ on $\mathcal{O}_2$ by $\zeta_1\varphi_1(\cdot)\zeta_1^* + \zeta_2\varphi_2(\cdot)\zeta_2^*$. Let $\xi \equiv (s_1, s_2)$, $\xi' \equiv ((s_1+s_2)/\sqrt{2},\ (s_1-s_2)/\sqrt{2}) \in \mathcal{O}_2 \times \mathcal{O}_2$. Then both $\varphi_1 +_\xi \varphi_2$ and $\varphi_1 +_{\xi'} \varphi_2$ are endomorphisms of $\mathcal{O}_2$ for any φ_1, φ_2. By using this notation, we can verify the following [16]:

$$\begin{cases} \psi_{14} = \alpha +_{\xi'} \alpha\theta, & \psi_{23} = \iota +_\xi \iota, & \psi_{123} = \iota +_\xi \alpha, \\ \psi_{124} = \alpha +_{\xi'} \alpha\beta_2, & \psi_{132} = \iota +_{\xi'} \beta_1, & \psi_{143} = \alpha +_{\xi'} \alpha\beta_1, \\ \psi_{234} = \iota +_{\xi'} \beta_2, & \psi_{1243} = \alpha +_\xi \alpha, & \psi_{1342} = \iota +_{\xi'} \theta, \end{cases} \tag{3.2}$$

where ι is the identity map on $\mathcal{O}_2$, $\alpha, \beta_1, \beta_2 \in \mathrm{Aut}\mathcal{O}_2$ are defined by $\alpha(s_i) \equiv s_{3-i}$, $\beta_j(s_i) \equiv (-1)^{\delta_{ij}} s_i$ for $i, j = 1, 2$ and $\theta \equiv \beta_1 \circ \beta_2$. On the other hand,

the following holds from Definition 2.1 (ii):

$$GP(+) \circ \beta_2 = GP(-), \quad GP(+) \circ \alpha = GP(+). \tag{3.3}$$

There are 16 unitary equivalence classes in $E_{2,2}$ by [15]. We choose 16 representatives in $E_{2,2}$ and show their branching laws in Table 2 by using Table II in [13], (3.2) and (3.3).

Table 2. Branching laws for $E_{2,2}$ on $\mathcal{O}_2$.

ψ_σ	$P(1) \circ \psi_\sigma$	$P(2) \circ \psi_\sigma$	$P(12) \circ \psi_\sigma$	$GP(+) \circ \psi_\sigma$
ψ_{id}	$P(1)$	$P(2)$	$P(12)$	$GP(+)$
$\psi_{(12)(34)}$	$P(2)$	$P(1)$	$P(12)$	$GP(+)$
ψ_{12}	$P(12)$	$P(1) \oplus P(2)$	$P(1122)$	—
ψ_{13}	$P(2)$	$P(2)$	$P(11)$	—
ψ_{24}	$P(1)$	$P(1)$	$P(22)$	—
ψ_{34}	$P(1) \oplus P(2)$	$P(12)$	$P(1122)$	—
ψ_{142}	$P(12)$	$P(12)$	$P(11) \oplus P(22)$	—
ψ_{14}	$P(22)$	$P(11)$	$P(12) \oplus P(12)$	$GP(+) \oplus GP(+) \circ \theta$
ψ_{23}	$P(1) \oplus P(1)$	$P(2) \oplus P(2)$	$P(12) \oplus P(12)$	$GP(+) \oplus GP(+)$
ψ_{123}	$P(1) \oplus P(2)$	$P(1) \oplus P(2)$	$P(12) \oplus P(12)$	$GP(+) \oplus GP(+)$
ψ_{124}	$P(22)$	$P(1) \oplus P(1)$	$P(1212)$	$GP(+) \oplus GP(-)$
ψ_{132}	$P(11)$	$P(2) \oplus P(2)$	$P(1212)$	$GP(+) \oplus GP(-) \circ \theta$
ψ_{143}	$P(2) \oplus P(2)$	$P(11)$	$P(1212)$	$GP(+) \oplus GP(-) \circ \theta$
ψ_{234}	$P(1) \oplus P(1)$	$P(22)$	$P(1212)$	$GP(+) \oplus GP(-)$
ψ_{1243}	$P(2) \oplus P(2)$	$P(1) \oplus P(1)$	$P(12) \oplus P(12)$	$GP(+) \oplus GP(+)$
ψ_{1342}	$P(11)$	$P(22)$	$P(12) \oplus P(12)$	$GP(+) \oplus GP(+) \circ \theta$

The part "—" is omitted because it is complicated and it is not necessary to classify ψ_σ's in this paper.

3.2 On UHF_2

The restriction of each of ψ_σ's in Table 1 on UHF_2 is also an endomorphism of UHF_2. We denote it by the same symbol for simplicity of description. For representations π and $\pi^{'}$ of a C*-algebra $\mathcal{A}$, we denote $\pi^{'} \prec \pi$ if $\pi^{'}$ is equivalent to a subrepresentation of π.

Lemma 3.1

$$P[12] \circ \psi_{12} = P[1122] \oplus P[2211], \quad P[21] \circ \psi_{12} = P[1221] \oplus P[2112],$$
$$P[12] \circ \psi_{13} = P[21] \circ \psi_{13} = P[1]. \tag{3.4}$$

Proof. By Table 2, we have $P(12) \circ \psi_{12} = P(1122)$. On the other hand, by Theorem 1.3, we have $P(12)|_{UHF_2} = P[12] \oplus P[21]$ and $P(1122)|_{UHF_2} = P[1122] \oplus P[2211] \oplus P[1221] \oplus P[2112]$. Let $(\mathcal{H}, \pi)$ be $P(12)$ with the GP

vector Ω. We denote $\pi(s_i)$ by s_i for simplicity of description. By defining $V_1 \equiv \overline{\pi(UHF_2)\Omega}$ and $V_2 \equiv \overline{\pi(UHF_2)s_2\Omega}$, we see that $(V_1, \pi|_{UHF_2})$ and $(V_2, \pi|_{UHF_2})$ are $P[12]$ and $P[21]$ with GP vectors $\Omega, s_2\Omega$ respectively. Let $e_1 \equiv \Omega$, $e_2 \equiv s_2\Omega$ and $t_i \equiv \psi_{12}(s_i)$ for $i = 1, 2$. Then we have $t_{1122}e_1 = e_1$, $t_{2112}e_2 = e_2$, $t_{1221}s_1e_1 = s_1e_1$, $t_{2211}s_2e_2 = s_2e_2$. From $e_1, s_2^2\Omega = s_2e_2 \in V_1$ and $e_2, s_1e_1 \in V_2$, we obtain $P[1122], P[2211] \prec V_1$ and $P[2112], P[1221] \prec V_2$. Hence, by using $(P[12] \oplus P[21]) \circ \psi_{12} = P[1122] \oplus P[1221] \oplus P[2211] \oplus P[2112]$, the statement for ψ_{12} holds.

Next, by Table 2 and Theorem 2.2 (iii), we have $P(12) \circ \psi_{13} = P(11) = P(1;+1) \oplus P(1;-1)$. From this result and Theorem 1.3, we obtain $(P[12] \oplus P[21]) \circ \psi_{13} = P[1] \oplus P[1]$. Therefore, the statement for ψ_{13} holds. ∎

Lemma 3.2 *On UHF_2, the following holds.*

(i) *$\psi_{13}, \psi_{12}, \psi_{24}$ and ψ_{34} are irreducible and proper.*

(ii) *ψ_{142} is reducible.*

Proof. (i) From $\alpha \circ \psi_{13} = \psi_{13}$, $\psi_{13}(UHF_2)$ is a subset of the fixed-point subalgebra $(UHF_2)^\alpha$ with respect to α. Hence, the image of ψ_{13} is a proper subset of UHF_2. On the other hand, from (3.4), ψ_{13} is irreducible. Therefore the statement for ψ_{13} holds.

Define the automorphism ϕ of $\mathcal{O}_2$ by $\phi(s_1) \equiv (s_1 + s_2)/\sqrt{2}$, $\phi(s_2) \equiv (-s_1 + s_2)/\sqrt{2}$. Then we have

$$\psi_{12} = (\mathrm{Ad}(\phi \circ \alpha))(\psi_{13}),\ \psi_{24} = \psi_{13} \circ \alpha,\ \psi_{34} = (\mathrm{Ad}(\alpha \circ \phi \circ \alpha))(\psi_{13}). \tag{3.5}$$

Therefore, using that $\phi|_{UHF_2} \in \mathrm{Aut}UHF_2$, the statements for ψ_{12}, ψ_{24} and ψ_{34} hold from that for ψ_{13}.
(ii) Let $\rho \equiv \psi_{142}$. By the inductive method, we see that for any $x \in UHF_2$, there exist $y, z \in UHF_2$ such that $\rho(x) = s_1ys_1^* + s_2zs_2^*$. For $a, b \in \mathbf{C}$, let $T_{a,b} \equiv aE_{11} + bE_{22} \in UHF_2$. Then we see that $T_{a,b}\rho(x) = \rho(x)T_{a,b}$ for any $x \in UHF_2$. Hence, $T_{a,b} \in \rho(UHF_2)' \cap UHF_2$ for each a, b. Therefore the statement holds. ∎
Because the unitary equivalence in $E_{2,2}$ in Table 1 is given by the unitary in UHF_2, there are at most 16 unitary equivalence classes in $UE_{2,2}$ in Theorem 1.4 with representatives in Table 2. Furthermore, we can verify the following identities in UHF_2:

$$\psi_{14} = \psi_{1243}, \quad \psi_{124} = \psi_{143}, \quad \psi_{132} = \psi_{234}, \quad \psi_{23} = \psi_{1342}. \tag{3.6}$$

Hence, there are at most 12 unitary equivalence classes in $UE_{2,2}$ with representatives as follows:

$$\left\{\psi_\sigma : \sigma = \begin{array}{l} id,\ (12)(34),\ (12),\ (13),\ (24),\ (34), \\ (142),\ (123),\ (14),\ (124),\ (132),\ (23) \end{array}\right\}. \tag{3.7}$$

From Table 2, Proposition 2.5, (3.2) and Lemmas 3.1, 3.2, we obtain Table 3.

Table 3. Branching laws for $UE_{2,2}$.

ψ_σ	$P[1]\circ\psi_\sigma$	$P[2]\circ\psi_\sigma$	$P[12]\circ\psi_\sigma$	$GP[+]\circ\psi_\sigma$	property
ψ_{id}	$P[1]$	$P[2]$	$P[12]$	$GP[+]$	*inn.aut*
$\psi_{(12)(34)}$	$P[2]$	$P[1]$	$P[21]$	$GP[+]$	*out.aut*
ψ_{12}	$P[12]\oplus P[21]$	$P[1]\oplus P[2]$	$P[1122]\oplus P[2211]$	—	*irr.end*
ψ_{13}	$P[2]$	$P[2]$	$P[1]$	—	*irr.end*
ψ_{24}	$P[1]$	$P[1]$	$P[2]$	—	*irr.end*
ψ_{34}	$P[1]\oplus P[2]$	$P[12]\oplus P[21]$	$P[1221]\oplus P[2112]$	—	*irr.end*
ψ_{142}	$P[12]\oplus P[21]$	$P[12]\oplus P[21]$	$P[1]\oplus P[2]$	—	*red.end*
ψ_{14}	$P[2]\oplus P[2]$	$P[1]\oplus P[1]$	$P[21]\oplus P[21]$	$GP[+]\oplus GP[+]$	*red.end*
ψ_{23}	$P[1]\oplus P[1]$	$P[2]\oplus P[2]$	$P[12]\oplus P[12]$	$GP[+]\oplus GP[+]$	*red.end*
ψ_{123}	$P[1]\oplus P[2]$	$P[1]\oplus P[2]$	$P[12]\oplus P[21]$	$GP[+]\oplus GP[+]$	*red.end*
ψ_{124}	$P[2]\oplus P[2]$	$P[1]\oplus P[1]$	$P[21]\oplus P[21]$	$GP[+]\oplus GP[-]$	*red.end*
ψ_{132}	$P[1]\oplus P[1]$	$P[2]\oplus P[2]$	$P[12]\oplus P[12]$	$GP[+]\oplus GP[-]$	*red.end*

In order to show formulae of $GP[+]\circ\psi_\sigma$, we use formulae $GP[\pm]\circ\theta = GP[\pm]$ $GP[+]\circ\beta_1 = GP[+]\circ\beta_2 = GP[-]$, and Proposition 2.7 (ii), (3.2) and Table 2.

Proof of Theorem 1.4. From (3.6) and $\#E_{2,2} = 24$, we have $\#UE_{2,2} \le 20$. (ii) is verified by using the latter statements of (1.6) with a direct computation on E_{JK}'s. Therefore, we obtain $\#(UE_{2,2}\setminus UG_2) \le 16$. We classify them with respect to the image $\psi_\sigma(s_1s_1^*)$ of ψ_σ at $s_1s_1^* \in UHF_2$ in Table 4.

Table 4. The image $\psi_\sigma(s_1s_1^*)$ of ψ_σ at $s_1s_1^*$.

$\psi_\sigma(s_1s_1^*)$	ψ_σ	$\psi_\sigma(s_1s_1^*)$	ψ_σ
$s_1s_1^*$	ψ_{12}, ψ_{34}	$s_1s_2s_2^*s_1^* + s_2s_1s_1^*s_2^*$	ψ_{13}, ψ_{123},
$s_2s_2^*$	ψ_{1324}, ψ_{1423}		$\psi_{134}(\sim\psi_{142}), \psi_{1234}(\sim\psi_{24})$
$s_1s_2s_2^*s_1^* + s_2s_2s_2^*s_2^*$	ψ_{14}, ψ_{124}	$s_1s_1s_1^*s_1^* + s_2s_2s_2^*s_2^*$	ψ_{24}, ψ_{142},
$s_1s_1s_1^*s_1^* + s_2s_1s_1^*s_2^*$	ψ_{23}, ψ_{132}		$\psi_{243}(\sim\psi_{123}), \psi_{1432}(\sim\psi_{13})$

Because, from Table 1 and Table 3, endomorphisms in each case are not unitarily equivalent, they are different as elements in $UE_{2,2}$. In consequence, we obtain that $\#(UE_{2,2}\setminus UG_2) = 16$. This implies that $\#UE_{2,2} = 20$ in (i). The latter in (i) follows from Table 3. Likewise, (iii) also holds from Table 3. ∎

We see that for any two endomorphisms in Table 3, their branching laws are different.

Notice: We summarize remarkable results in this subsection. (i) There is a proper irreducible endomorphism of $\mathcal{O}_2$ such that its restriction on UHF_2 is not irreducible (see ψ_{142} in Table 1 and Table 3). (ii) There is an endomorphism ρ of UHF_2 such that there are two different extensions of ρ to $\mathcal{O}_2$ and they are not equivalent in End$\mathcal{O}_2$ (compare Table 2 and (3.6)). (iii) Table 3 is the complete classification of endomorphisms in $UE_{2,2}$ with respect to unitary equivalence. Of course, this is properly finer than the classification by statistical dimension [5, 11]. Furthermore, we see that 4 equivalence classes of irreducible and proper endomorphisms in $UE_{2,2}$. In this sense, Table 3 contains nontrivial results.

3.3 Nakanishi endomorphism restricted on UHF_3

Our studies for endomorphisms were inspired by the following endomorphism ρ_ν of $\mathcal{O}_3$ discovered by Noboru Nakanishi:

$$\begin{cases} \rho_\nu(s_1) \equiv s_2s_3s_1^* + s_3s_1s_2^* + s_1s_2s_3^*, \\ \rho_\nu(s_2) \equiv s_3s_2s_1^* + s_1s_3s_2^* + s_2s_1s_3^*, \\ \rho_\nu(s_3) \equiv s_1s_1s_1^* + s_2s_2s_2^* + s_3s_3s_3^*, \end{cases} \tag{3.8}$$

where s_1, s_2, s_3 are canonical generators of $\mathcal{O}_3$. In Theorem 1.2 of [13], we proved that ρ_ν is irreducible and neither an automorphism nor equivalent to the canonical endomorphism of $\mathcal{O}_3$. By Table I (b) in [15], we obtained the following:

$$P(1) \circ \rho_\nu = P(3) \oplus P(12), \quad P(12) \circ \rho_\nu = P(113223). \tag{3.9}$$

Because ρ_ν is one of the second order permutative endomorphisms of $\mathcal{O}_3$, $\rho_\nu|_{UHF_3}$ is also an endomorphism of UHF_3.

Proposition 3.3

$$P[1] \circ \rho_\nu = P[3] \oplus P[12] \oplus P[21], \tag{3.10}$$
$$P[12] \circ \rho_\nu = P[113223] \oplus P[322311] \oplus P[231132], \tag{3.11}$$
$$P[21] \circ \rho_\nu = P[223113] \oplus P[311322] \oplus P[132231]. \tag{3.12}$$

Proof. The first equation holds from (3.9) and $P(12)|_{UHF_3} = P[12] \oplus P[21]$. In the same way, we have

$$\begin{array}{rl}(P[12] \oplus P[21]) \circ \rho_\nu = & P[113223] \oplus P[322311] \oplus P[231132] \\ & \oplus P[223113] \oplus P[311322] \oplus P[132231].\end{array} \tag{3.13}$$

Let $(\mathcal{H}, \pi)$ be $P(12)$ of $\mathcal{O}_3$ with the GP vector Ω. Then $e_1 \equiv \Omega$ and $e_2 \equiv s_2\Omega$ are GP vectors with respect to $P[12]$ and $P[21]$ in $\mathcal{H}$, respectively. Define $V_1 \equiv \overline{\pi(UHF_3)e_1}$ and $V_2 \equiv \overline{\pi(UHF_3)e_2}$. Then $e_1, e_2, s_1e_1, s_3e_1, s_2e_2, s_3e_2$ are elements of the cycle of $P(113223)$ in $\mathcal{H}$. We see that $e_1 \in V_1, s_2e_2 = s_2s_2e_1 = s_{22,21}e_1 \in V_1, s_3e_2 = s_{32,21}e_1 \in V_1$, $e_2 \in V_2, s_1e_1 = s_2s_1e_2 = s_{21,12}e_2 \in V_2, s_3e_1 = s_{31,12}e_2 \in V_2$. Let $t_i \equiv \rho_\nu(s_i)$ for $i = 1, 2, 3$. Then we have $t_{113223}e_1 = e_1$, $t_{223113}e_2 = e_2$, $t_{311322}s_1e_1 = s_1e_1$, $t_{132231}s_3e_1 = s_3e_1$, $t_{322311}s_2e_2 = s_2e_2$ and $t_{231132}s_3e_2 = s_3e_2$. Therefore, we obtain $P[113223], P[322311], P[231132] \prec V_1$ and $P[223113], P[311322], P[132231] \prec V_2$. From (3.13), the statement holds. ∎

4 Branching laws on fermions

4.1 Fock representation and infinite wedge representation

The Fock representation and the infinite wedge representation are well-known representations of fermions and they are important not only in physics but also in mathematics [17, 18]. However there are few studies as a representation theory of the fermion algebra itself. We review a relation between them and permutative representations [14].

Definition 4.1 (i) *The Fock representation of $\mathcal{A}$ is the class of representation $(\mathcal{H}, \pi)$ with a cyclic vector $\Omega \in \mathcal{H}$ such that*

$$\pi(a_n)\Omega = 0 \quad (\textit{for all } n \in \mathbf{N}). \tag{4.1}$$

(ii) *The infinite wedge representation of $\mathcal{A}$ is the class of representation $(\mathcal{H}, \pi)$ with a cyclic vector $\Omega \in \mathcal{H}$ such that*

$$b_{-k}\Omega = b_k^*\Omega = 0 \quad (\textit{for all } k \in \mathbf{Z}_{\geq} + 1/2) \tag{4.2}$$

where $b_k \equiv \pi(a_{2k+1})$, $\quad b_{-k} \equiv \pi(a_{2k})$ for $k \in \mathbf{Z}_{\geq} + 1/2$.

(iii) *$(\mathcal{H}^*, \pi^*)$ is the dual of $(\mathcal{H}, \pi)$ if $(\mathcal{H}^*, \pi^*)$ is equivalent to $(\mathcal{H}, \pi \circ \varphi)$ where φ is the $*$-automorphism of $\mathcal{A}$ defined by $\varphi(a_n) \equiv (-1)^{n-1}a_n^*$ for each n.*

Notice: (a) The Fock representation and the infinite wedge representation are different from each other not only in numbering of generators but also in roles of creations and annihilations. The infinite wedge representation is often called as the Fock representation in a broad sense. In this paper, we distinguish them. (b) The map φ in Definition 4.1 (iii) is defined linearly but not conjugate linearly. Hence, $(\mathcal{H}^*, \pi^*)$ is not a conjugate representation of $(\mathcal{H}, \pi)$. The naming "dual" is in conformity with the dual infinite wedge in [17].

In Definition 4.1 (i) and (ii), Ω in both cases is called the *vacuum vector*. The creation and annihilation (operator) with respect to their representations and vacua are given in Table 5.

Table 5. Creations and anihilations.

	$Fock$	$Fock^*$	IW	IW^*
creation	a_n^*	a_n	$a_{2n-1}^*,\ a_{2n}$	$a_{2n-1},\ a_{2n}^*$
annihilation	a_n	a_n^*	$a_{2n-1},\ a_{2n}^*$	$a_{2n-1}^*,\ a_{2n}$

Here $n \in \mathbf{N}$, and $Fock, Fock^*, IW$ and IW^* stand for the Fock representation, the dual Fock representation, the infinite wedge representation and the dual infinite wedge representation of $\mathcal{A}$, respectively.

We regard $\mathcal{A} = UHF_2$ as the same subalgebra of $\mathcal{O}_2$ by means of maps Φ_{CAR}, Φ_{UHF_2}, Ψ in (1.2), (1.4), (1.7), respectively. Restrictions of representations and branching laws are also described by such identifications.

Theorem 4.2

(i) *The (dual) Fock representation is* $P[1]$ *(resp.* $P[2]$*).*

(ii) *The (dual) infinite wedge representation is* $P[12]$ *(resp.* $P[21]$*).*

(iii) *In both (i) and (ii), the vacuum vector of the former is the GP vector of the latter up to scalar multiple.*

Proof. (i) In [1], we denoted $P(1)$ by Rep(1). From $P(1)|_{UHF_2} = P(1)|_{CAR} = P[1]$ and § 3.3 of [1], the Fock representation is $P[1]$. From this result and $P[2] \circ \alpha = P[1]$, the statement holds.
(ii) This is shown in Proposition 3.6 (ii) of [14].
(iii) Because of the uniqueness of the vacuum and GP vectors, this is shown according to identification in (1.7). ∎

By Theorem 4.2, the four simplest examples $P[1], P[2], P[12], P[21]$ of permutative representations of UHF_2 are interpreted as well-known four representations in physics. In this sense, the isomorphism Ψ, embeddings

Φ_{CAR} and Φ_{UHF_2} are compatible with permutative representations of $\mathcal{O}_2$ and UHF_2, and they acquire importance in mathematical physics.

4.2 Permutative endomorphisms restricted on fermions

We show formulae of the second order permutative endomorphisms restricted on the CAR algebra $\mathcal{A}$. We explain the computation method by using an example in $E_{2,2}$.

Lemma 4.3 *For $n \geq 1$, the following holds:*

$$\psi_{142}(a_{2n-1}) = (-1)^{n-1}(a_1 a_1^* a_{2n} - a_1^* a_1 a_{2n}^*), \tag{4.3}$$

$$\psi_{142}(a_{2n}) = (-1)^{n-1}(a_1 a_1^* a_{2n+1}^* + a_1^* a_1 a_{2n+1}). \tag{4.4}$$

Proof. Let $\rho \equiv \psi_{142}$. First, we see that $\rho(a_1) = s_1 a_1 s_1^* + s_2 a_1^* s_2^*$ and $\rho(a_2) = s_1 a_2^* s_1^* - s_2 a_2 s_2^*$. Next, we can verify that if X and Y in $\mathcal{O}_2$ satisfy that $\rho(X) = s_1 X s_1^* + s_2 X^* s_2^*$ and $\rho(Y) = s_1 Y^* s_1^* - s_2 Y s_2^*$, then $\rho(\zeta(X)) = s_1\zeta(X)^* s_1^* - s_2\zeta(X)s_2^*$ and $\rho(\zeta(Y)) = -s_1\zeta(Y)s_1^* - s_2\zeta(Y)^* s_2^*$ where ζ is defined in (1.3). By the induction method, we obtain that $\rho(a_{2n-1}) = (-1)^{n-1}(s_1 a_{2n-1} s_1 + s_2 a_{2n-1}^* s_2^*)$ and $\rho(a_{2n}) = (-1)^{n-1}(s_1 a_{2n}^* s_1 - s_2 a_{2n} s_2^*)$. Since we have $s_1 X s_1^* = a_1 a_1^* \zeta(X)$ and $s_2 X s_2^* = -a_1^* a_1 \zeta(X)$ for any $X \in \mathcal{O}_2$, the statements hold. ∎

In this way, we compute $\psi_\sigma(a_n)$ for every element in Table 3 and we show their properties in Table 6.

Table 6. Elements in $UE_{2,2}$ on fermions.

ψ_σ	$\psi_\sigma(a_n)$	property
ψ_{id}	a_n	*inn.aut*
$\psi_{(12)(34)}$	$a_1 \quad (n=1), \quad (-1)^n a_n^* \quad (n \geq 2)$	*out.aut*
ψ_{12}	—	*irr.end*
ψ_{13}	—	*irr.end*
ψ_{24}	—	*irr.end*
ψ_{34}	—	*irr.end*
ψ_{142}	$(-1)^{k-1}(a_1 a_1^* a_{2k} - a_1^* a_1 a_{2k}^*) \quad (n = 2k-1)$	*red.end*
	$(-1)^{k-1}(a_1 a_1^* a_{2k+1}^* + a_1^* a_1 a_{2k+1}) \quad (n = 2k)$	
ψ_{14}	$(-1)^{n-1}(a_1 a_1^* - a_1^* a_1)a_{n+1}^*$	*red.end*
ψ_{23}	$(a_1 a_1^* - a_1^* a_1)a_{n+1}$	*red.end*
ψ_{123}	$a_1 a_1^* a_{n+1} + (-1)^n a_1^* a_1 a_{n+1}^*$	*red.end*
ψ_{124}	$(-1)^n(a_1^* - a_1)a_{n+1}^*$	*red.end*
ψ_{132}	$(a_1^* - a_1)a_{n+1}$	*red.end*

Because $\psi_{12}, \psi_{13}, \psi_{24}$ and ψ_{34} are irreducible and proper, they are important as nontrivial endomorphisms (or sectors) of $\mathcal{A}$. We show their first three formulae in Table 7.

Table 7. $\psi_\sigma(a_1), \psi_\sigma(a_2), \psi_\sigma(a_3)$ for $\sigma = (12), (13), (24), (34)$.

$\psi_{12}(a_1) =$	$-a_1(a_2 + a_2^*)$
$\psi_{12}(a_2) =$	$-(a_1a_1^*a_2^* + a_1^*a_1a_2)(a_3 + a_3^*)$
$\psi_{12}(a_3) =$	$\{a_1a_1^*(a_2a_2^*a_3 + a_2^*a_2a_3^*) - a_1^*a_1(a_2^*a_2a_3 + a_2a_2^*a_3^*)\}(a_4 + a_4^*)$
$\psi_{13}(a_1) =$	$a_1^*a_2a_2^* + a_1a_2^*a_2$
$\psi_{13}(a_2) =$	$(a_1^* + a_1)(-a_2^*a_3a_3^* + a_2a_3^*a_3)$
$\psi_{13}(a_3) =$	$(a_1^* - a_1)(-a_2^* + a_2)(-a_3^*a_4a_4^* + a_3a_4^*a_4)$
$\psi_{24}(a_1) =$	$a_1a_2a_2^* + a_1^*a_2^*a_2$
$\psi_{24}(a_2) =$	$-(a_1^* + a_1)(-a_2a_3a_3^* + a_2^*a_3^*a_3)$
$\psi_{24}(a_3) =$	$(a_1^* - a_1)(-a_2^* + a_2)(-a_3a_4a_4^* + a_3^*a_4^*a_4)$
$\psi_{34}(a_1) =$	$-a_1(a_2 + a_2^*)$
$\psi_{34}(a_2) =$	$-(a_1a_1^*a_2 + a_1^*a_1a_2^*)(a_3 + a_3^*)$
$\psi_{34}(a_3) =$	$\{-a_1a_1^*(a_2a_2^*a_3 + a_2^*a_2a_3^*) + a_1^*a_1(a_2^*a_2a_3 + a_2a_2^*a_3^*)\}(a_4 + a_4^*)$

For Table 7, we use $\psi_{13} \circ \alpha = \psi_{24}$, $\alpha \circ \psi_{12} \circ \alpha = \psi_{34}$ and $\alpha(a_n) = (-1)^{n-1}a_n^*$. From Table 3 and Theorem 4.2, we obtain branching laws on $\mathcal{A}$ in Table 8.

Table 8. Branching laws restricted for $UE_{2,2}$ on fermions.

ψ_σ	$Fock \circ \psi_\sigma$	$Fock^* \circ \psi_\sigma$	$IW \circ \psi_\sigma$
ψ_{id}	$Fock$	$Fock^*$	IW
$\psi_{(12)(34)}$	$Fock^*$	$Fock$	IW^*
ψ_{12}	$IW \oplus IW^*$	$Fock \oplus Fock^*$	$P[1122] \oplus P[2211]$
ψ_{13}	$Fock^*$	$Fock^*$	$Fock$
ψ_{24}	$Fock$	$Fock$	$Fock^*$
ψ_{34}	$Fock \oplus Fock^*$	$IW \oplus IW^*$	$P[1221] \oplus P[2112]$
ψ_{142}	$IW \oplus IW^*$	$IW \oplus IW^*$	$Fock \oplus Fock^*$
ψ_{14}	$Fock^* \oplus Fock^*$	$Fock \oplus Fock$	$IW^* \oplus IW^*$
ψ_{23}	$Fock \oplus Fock$	$Fock^* \oplus Fock^*$	$IW \oplus IW$
ψ_{123}	$Fock \oplus Fock^*$	$Fock \oplus Fock^*$	$IW \oplus IW^*$
ψ_{124}	$Fock^* \oplus Fock^*$	$Fock \oplus Fock$	$IW^* \oplus IW^*$
ψ_{132}	$Fock \oplus Fock$	$Fock^* \oplus Fock^*$	$IW \oplus IW$

From Table 8, the Fock representation, the infinite wedge representation and their duals are transformed to each other by endomorphisms. In case a branching occurs, new vacua appear instead of the original vacuum. For example, we see that (1.8) is nothing but $\psi_{142}|_{\mathcal{A}}$. The branching law $Fock \circ \psi_{142} = IW \oplus IW^*$ means the Fock vacuum is transformed to two vacua, that is, the infinite wedge vacuum and the dual infinite wedge vacuum, by ψ_{142}.

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